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***Rapporto Tecnico N.: 1***
**RT-ICAR-PA-09-01** **Marzo 2009**

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**Abstract**

In this paper a numerical method avoiding mesh generation is proposed. The method performs an integral representation in modelling the solution by means of a smoothing kernel function. Appreciable numerical results are gained if suitable conditions are imposed on the smoothing kernel function. In the discrete formulation the method involves domain particles and sometimes the kernel conditions are loosen. A fundamental task in a meshless framework is the kernel conditions restoring, usually named consistency. In this paper the meshless SPH method is used and studies on consistency restoring are carried out. Various tests are performed approximating 1D, 2D and 3D trigonometric functions with even and uneven data set achieving good numerical results.



*Key words: meshless particle method, Smoothed Particle Hydrodynamics method, consistency restoring, function approximation*


## 1. Introduction

The meshless methods are a valid computational alternative to grid methods in the simulation of a wide problems variety. They share common features such as the avoidance of the use of grids, but are different in functions approximation and computational processes ([4]).
Smoothed Particle Hydrodynamics (SPH) ([9], [12], [15], [16], [17]) is a meshless method and its attractiveness and popularity is due to the evaluation of unknown field functions and relative differential operators by means of an integral representation based on a suitable interpolating function ([10]). The integral representation is discretized by using a set of *particles* scattered in the problem domain, making the methods intrinsically adaptive.
In simulating physical phenomena, it is often necessary to control regions with high localized field gradient or to better reproduce irregular geometries of the problem domain. In these cases an uneven particles distribution has to be considered into the problem domain. This occurrence can lead to a lack of consistency of the method so that modified formulations have to be adopted to improve the result. Different corrective strategies can be performed: corrections can be provided through suitable transformations or on approximating functions, and consequently on their derivatives, or directly on derivatives of approximations without restoring the consistency of the approximating functions ([1], [2], [3], [5], [6], [11], [13]).
In this paper, the particle inconsistency problem is investigated and the approach proposed in [13] is applied to restore the particle consistency in approximating functions. In order to show the capability of the proposed methodology, simulations with even and uneven particles distributions are performed.
The paper is organized as follows: in section 2 the background of SPH method is presented and an analysis of the fundamental issues related to the consistency restoring techniques is reported; in section 3 numerical experiments are carried out on 1D, 2D and 3D trigonometric functions. Finally section 4 closes the paper by proposing future works.

## 2. Smoothed Particle Hydrodynamics Method Fundamentals

In order to approximate a function $f(\boldsymbol{x})$ in a domain $\Omega \subseteq \mathrm{R}^d$ a meshless method, as the SPH method, works initially involving a function $W$ employed to define the *kernel approximation* of $f$:

$$f^h(\boldsymbol{x}) = \int_\Omega f(\boldsymbol{y}) W(\boldsymbol{x}-\boldsymbol{y},h) d\boldsymbol{y} \,. \tag{1}$$

The function *W* is called the *smoothing kernel function* depending on the spatial variables and on the *smoothing length* parameter *h*:

$$W(\boldsymbol{x}-\boldsymbol{y},h) = \alpha_d K(\mathrm{R}) \,, \tag{2}$$

where $\mathrm{R} = \|\boldsymbol{x}-\boldsymbol{y}\| / h$ and $\alpha_d$ is a dimension-dependent normalization constant. Many kinds of smoothing kernel functions have shown in SPH literature, the most popular are the bell-shaped functions ([8], [9], [12], [14], [15], [17]). The smoothing kernel function should satisfy the following conditions:

a) *delta Dirac function condition*: $\lim_{h\to 0} W(s,h) = \delta(s)$ where $\delta(s)$ is the Dirac delta function and $s = Rh$;
b) *compactness condition*: $W(\boldsymbol{x}-\boldsymbol{y},h) > 0$ on a subdomain of $\Omega$ and $W(\boldsymbol{x}-\boldsymbol{y},h) = 0$ outside the subdomain;
c) *normalization condition*: $\int_\Omega W(\boldsymbol{x}-\boldsymbol{y},h) d\boldsymbol{y} = 1$;
d) *monotonic condition*: $W(s,h)$ is a monotonically decreasing function;
e) *symmetric condition*: $W$ is an even function $\int_\Omega (\boldsymbol{x}-\boldsymbol{y}) W(\boldsymbol{x}-\boldsymbol{y},h) d\boldsymbol{y} = 0$.

By introducing a number of points (or *particles*) arbitrarily distributed to cover the problem domain, the kernel approximation can be discretized. The compactness condition (b) means that only a finite number of particles referred as *nearest neighboring particles* (NNP) have to be considered for a satisfactory approximation. Therefore, kernel approximation is discretized by summing the contribution over all the NNP ([9], [12], [15], [16]) obtaining the so called *particle approximation*. Thus, the particle approximation of a function *f* is obtained by averaging function values $f(\boldsymbol{x}_j)$ involving all NNP of $\boldsymbol{x}$:

$$f^h(\boldsymbol{x}) \cong \sum_{j=1}^{N} f(\boldsymbol{x}_j) W(\boldsymbol{x}-\boldsymbol{x}_j,h) V_j \,, \tag{3}$$

where $V_j$ is the measure of the support domain surrounding the particle $\boldsymbol{x}_j$. The smoothing length *h* and the number of particles determine the resolution of the approximation, so, a crucial task before performing any computation using the SPH method is the NNP spotting which is of primary importance by working with meshless methods.
The particle approximation of a function *f* can be affected by some numerical problems ([12]) such as the particle inconsistency ([3], [18]) which can lead to low approximation accuracy. In effect, the consistency conditions of the kernel approximation c) and e) could not ensure the consistency conditions in the discrete formulation, i.e.:

$$\sum_{j=1}^{N}(\boldsymbol{x}-\boldsymbol{x}_j)^k W(\boldsymbol{x}-\boldsymbol{x}_j,h)V_j=\delta_{k0}, \qquad k=0,1. \tag{4}$$

The consistency conditions for SPH approximation can be expressed as its ability to exactly reproduce a polynomial up to the *k*-th order so that the approximation is said to have the *k*-th order of consistency ([3], [12], [14]). The particle inconsistency originates from the discrepancy between the SPH kernel and particle approximations: boundary particles, irregular distributed particles and variable smoothing length can usually produce inconsistency in the particle approximation process ([12]). In simulating many classes of problems, it is often necessary to better reproduce irregular geometries of the problem domain or to control regions with high localized field gradient, large deformations and moving discontinuities. Attempts to insert and/or remove particles, where it is necessary, inevitably lead to an uneven spacing, making this issue extremely critical. An uneven particle spacing usually yields to a lack of consistency, so that different strategies have to be developed to restore the particle consistency. Consistency can be restored through corrective transformations and different modified SPH formulations can be performed ([1], [2], [3], [5], [6], [11], [13]). Many approaches in consistency restoring work by modifying kernel shape ([7]). The corresponding particle approximations often suffer to be not equal in consistency on all problem domain.
An interesting approach to restore the particle consistency, based on Taylor series expansion of $f(\boldsymbol{x})$ around $\boldsymbol{x}_j$, can be adopted to overcome this occurrence and by working with *k*-th order of particle consistency for both interior and boundary regions ([13]); moreover this technique is almost insensitive with respect to the particles distribution. The methodology approximates $f$ in $\Omega\subseteq \mathrm{R}^d$ as follows:

$$\begin{aligned}f(\boldsymbol{y})&=f(\boldsymbol{x})+(\boldsymbol{y}^r-\boldsymbol{x}^r)f_r(\boldsymbol{x})+\frac{(\boldsymbol{y}^r-\boldsymbol{x}^r)(\boldsymbol{y}^s-\boldsymbol{x}^s)}{2!}f_{rs}(\boldsymbol{x})+\ldots\\&+\frac{(\boldsymbol{y}^r-\boldsymbol{x}^r)(\boldsymbol{y}^s-\boldsymbol{x}^s)\cdots(\boldsymbol{y}^i-\boldsymbol{x}^i)}{M!}f_{\underbrace{rs\ldots i}_{M}}(\boldsymbol{x})+\ldots,\qquad r,s,\ldots,i,\ldots=1,\ldots,d.\end{aligned} \tag{5}$$

where $\boldsymbol{x}^\alpha$, $\alpha=r,s,\ldots,i,\ldots$ is the $\alpha$-component of $\boldsymbol{x}$, $f_\alpha(\boldsymbol{x})=\dfrac{\partial f}{\partial \boldsymbol{y}^\alpha}(\boldsymbol{x})$, $\alpha=r,s,\ldots,i,\ldots$, $f_{rs\ldots i}(\boldsymbol{x})=\dfrac{\partial f}{\partial \boldsymbol{y}^r}\dfrac{\partial f}{\partial \boldsymbol{y}^s}\ldots\dfrac{\partial f}{\partial \boldsymbol{y}^i}(\boldsymbol{x})$; next, both the terms of (5) are multiplied for $\nabla^k W(\boldsymbol{x}-\boldsymbol{y},h), k=0,1,\ldots$, and integrated on $\Omega\subseteq\mathrm{R}^d$:

$$\begin{aligned}&\int_\Omega f(\boldsymbol{y})\nabla^k W(\boldsymbol{x}-\boldsymbol{y},h)d\boldsymbol{y}=\\&=\int_\Omega f(\boldsymbol{x})\nabla^k W(\boldsymbol{x}-\boldsymbol{y},h)d\boldsymbol{y}+\int_\Omega(\boldsymbol{y}^r-\boldsymbol{x}^r)f_r(\boldsymbol{x})\nabla^k W(\boldsymbol{x}-\boldsymbol{y},h)d\boldsymbol{y}+\\&+\int_\Omega\frac{(\boldsymbol{y}^r-\boldsymbol{x}^r)(\boldsymbol{y}^s-\boldsymbol{x}^s)}{2!}f_{rs}(\boldsymbol{x})\nabla^k W(\boldsymbol{x}-\boldsymbol{y},h)d\boldsymbol{y}+\ldots\\&+\int_\Omega\frac{(\boldsymbol{y}^r-\boldsymbol{x}^r)(\boldsymbol{y}^s-\boldsymbol{x}^s)\cdots(\boldsymbol{y}^i-\boldsymbol{x}^i)}{M!}f_{\underbrace{rs\ldots i}_{M}}(\boldsymbol{x})\nabla^k W(\boldsymbol{x}-\boldsymbol{y},h)d\boldsymbol{y}+\ldots,\end{aligned} \tag{6}$$

with $r,s,\ldots,i,\ldots=1,\ldots,d$ and $k=0,1,\ldots$. The equations (6) can be discretized and expressed as a system $A{\cdot}b$=$c$ (7) where:

$A$=

$$\begin{pmatrix} \sum_{j=1}^{N}\nabla^0 W V_j & \sum_{j=1}^{N}(\boldsymbol{x}_j^r-\boldsymbol{x}^r)\nabla^0 W V_j & \cdots & \sum_{j=1}^{N}\frac{(\boldsymbol{x}_j^r-\boldsymbol{x}^r)(\boldsymbol{x}_j^s-\boldsymbol{x}^s)\ldots(\boldsymbol{x}_j^i-\boldsymbol{x}^i)}{M!}\nabla^0 W V_j & \cdots \\ \sum_{j=1}^{N}\nabla^1 W V_j & \sum_{j=1}^{N}(\boldsymbol{x}_j^r-\boldsymbol{x}^r)\nabla^1 W V_j & \cdots & \sum_{j=1}^{N}\frac{(\boldsymbol{x}_j^r-\boldsymbol{x}^r)(\boldsymbol{x}_j^s-\boldsymbol{x}^s)\ldots(\boldsymbol{x}_j^i-\boldsymbol{x}^i)}{M!}\nabla^1 W V_j & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ \sum_{j=1}^{N}\nabla^k W V_j & \sum_{j=1}^{N}(\boldsymbol{x}_j^r-\boldsymbol{x}^r)\nabla^k W V_j & \cdots & \sum_{j=1}^{N}\frac{(\boldsymbol{x}_j^r-\boldsymbol{x}^r)(\boldsymbol{x}_j^s-\boldsymbol{x}^s)\ldots(\boldsymbol{x}_j^i-\boldsymbol{x}^i)}{M!}\nabla^k W V_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$b=\begin{pmatrix} f(\boldsymbol{x}) \\ f_r(\boldsymbol{x}) \\ \vdots \\ f_{\underbrace{rs\ldots i}_{M}}(\boldsymbol{x}) \\ \vdots \end{pmatrix} \text{ and } c=\begin{pmatrix} \sum_{j=1}^{N} f(\boldsymbol{x}_j)\nabla^0 W(\boldsymbol{x}-\boldsymbol{x}_j,h)V_j \\ \sum_{j=1}^{N} f(\boldsymbol{x}_j)\nabla^1 W(\boldsymbol{x}-\boldsymbol{x}_j,h)V_j \\ \vdots \\ \sum_{j=1}^{N} f(\boldsymbol{x}_j)\nabla^k W(\boldsymbol{x}-\boldsymbol{x}_j,h)V_j \\ \vdots \end{pmatrix}$$

with $\nabla^k W=\nabla^k W(\boldsymbol{x}-\boldsymbol{x}_j,h)$, $r,s,\ldots,i,\ldots=1,\ldots,d$ and $k=0,1,\ldots$. By solving the linear system (7), function values and its derivatives are obtained with the $k$-th order of consistency kept on all over problem domain and no modifications on the smoothing kernel function are required. It's straightforward that to ensure a $k$-th order of consistency, $k$-th order derivatives have to be retained in Taylor series expansion.

## 3. Numerical Investigations

In this section the consistency restoring approach outlined above is used to numerically approximate a trigonometric function in 1D, 2D and 3D domains by ensuring a 1-st order consistency; this can be accomplished retaining only the 1-st order derivatives in the Taylor series expansion.

Computational results obtained are compared with the analytic functions profiles, the SPH approximations (without consistency restoring) and the approximations obtained by imposing the 0-th order consistency ([19]).

The well-known Gaussian function is chosen as smoothing kernel function:

$$W(\boldsymbol{x}-\boldsymbol{y},h)=\alpha_d\exp(-R^2), \tag{8}$$

where $\alpha_d$ equals $1/\pi^{1/2}h, 1/\pi h^2, 1/\pi^{3/2}h^3$ respectively in one, two and three dimensions.

First, the described method is applied to approximate 500 values of the 1D function $f(x)=\cos(x)$ in the interval $[0,2\pi]$ starting from 50 evenly, non-uniformly and randomly distributed particles, respectively. The uneven distribution is obtained in such a way that $V_j/V_{j-1}=1.1$ with $j=2,\ldots,N$ ([13]). In order to compare the

accuracies of the obtained approximations the root-mean-square errors have been calculated and presented in Table 1, while Figure 1 shows the obtained approximations.
In the second case, the 2D function $f(x, y) = \cos(x + y)$ in the interval $[0, 2\pi]\times[0, 2\pi]$ has been used approximating $50\times 50$ values of $f(x, y)$ starting from $20\times 20$ evenly, non-uniformly and randomly distributed particles, respectively. For the uneven distribution the same idea outlined above is used. Table 2 presents the obtained root-mean-square errors and Figure 2 shows the obtained space profiles.
Finally, in the third case, the 3D function $f(x, y, t) = \cos(x + y + t)$ in the interval $[0, 2\pi]\times[0, 2\pi]\times[0, 2\pi]$ has been used approximating $40\times 40\times 40$ values of $f(x, y, t)$ starting from $20\times 20\times 20$ evenly, non-uniformly and randomly distributed particles and Table 3 presents the obtained root-mean-square errors, while Figure 3 shows the obtained approximations.
Obtained results show a general improvement of the approximation quality when the 1-st order consistency is applied with respect to the 0-th order and the SPH without any consistency restoring.
For example, in the 1D test case, when the even and random distributions are used the 1-st order consistency gains from one to two orders of magnitude with respect to the SPH without any consistency restoring, while reduces from 1/4 to 1/10 the error with respect to the 0-th order of consistency. Surprisingly, when the uneven distribution is used, the obtained improvement is very small; this could be caused by the particular particle distribution with respect to the regularity properties of the chosen functions. This behavior has to be deeply investigated. Finally, 2D and 3D test cases show behaviors similar to the 1D test case, even though the quality improvement is lesser than the 1D one.

## 4. Conclusions and Future Work

In this paper a meshless SPH method has been presented and studies on consistency restoring have been carried out. Various numerical tests approximating 1D, 2D and 3D functions demonstrated the effectiveness of the proposed approach. Future works will be devoted to investigate the usage of the SPH framework in the Image Processing context.

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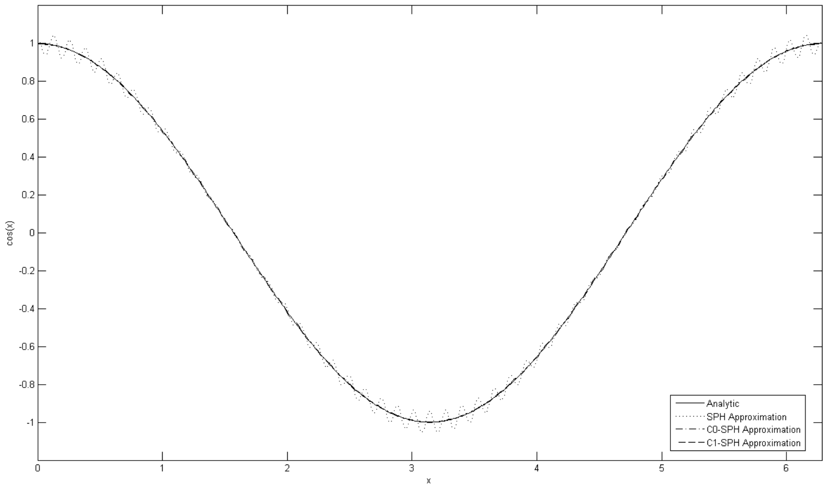
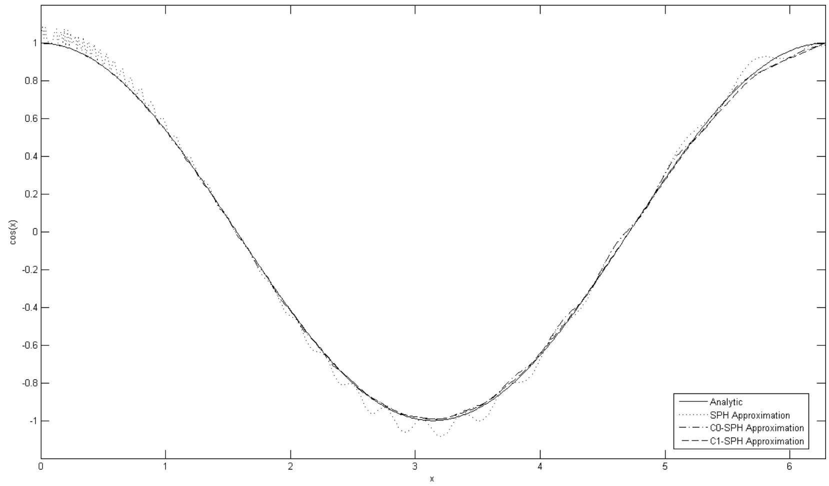
**(A)**




**(B)**




**(C)**

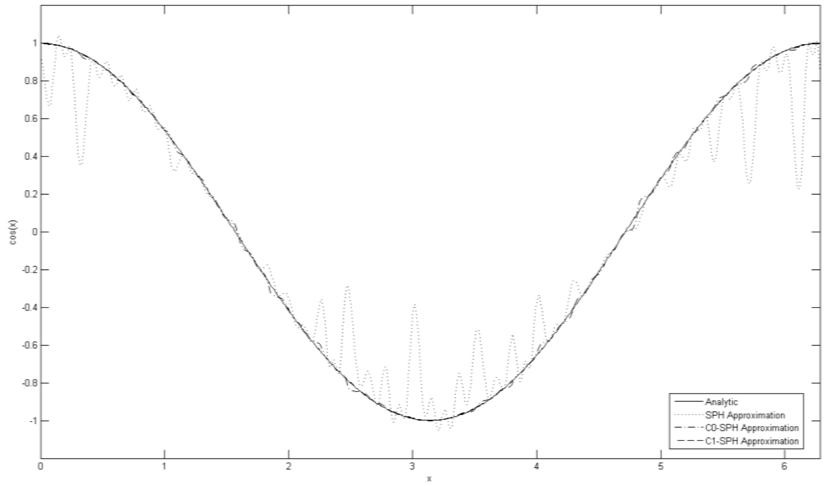



**Figure 1:** Comparison among the space profiles of the analytic 1D function $f(x) = \cos(x)$ and SPH simulations by using even (A), uneven (B) and random (C) distributions.

| **Algorithm** | **Even distribution** | **Uneven distribution** | **Random distribution** |
|---|---|---|---|
| No Consistency | 0.0610 | 0.0947 | 0.7574 |
| 0-th order consistency | 0.0085 | 0.0418 | 0.0461 |
| 1-st order consistency | 0.0021 | 0.0399 | 0.0034 |

**Table 1:** root-mean-square errors obtained for the 1D function $f(x) = \cos(x)$.

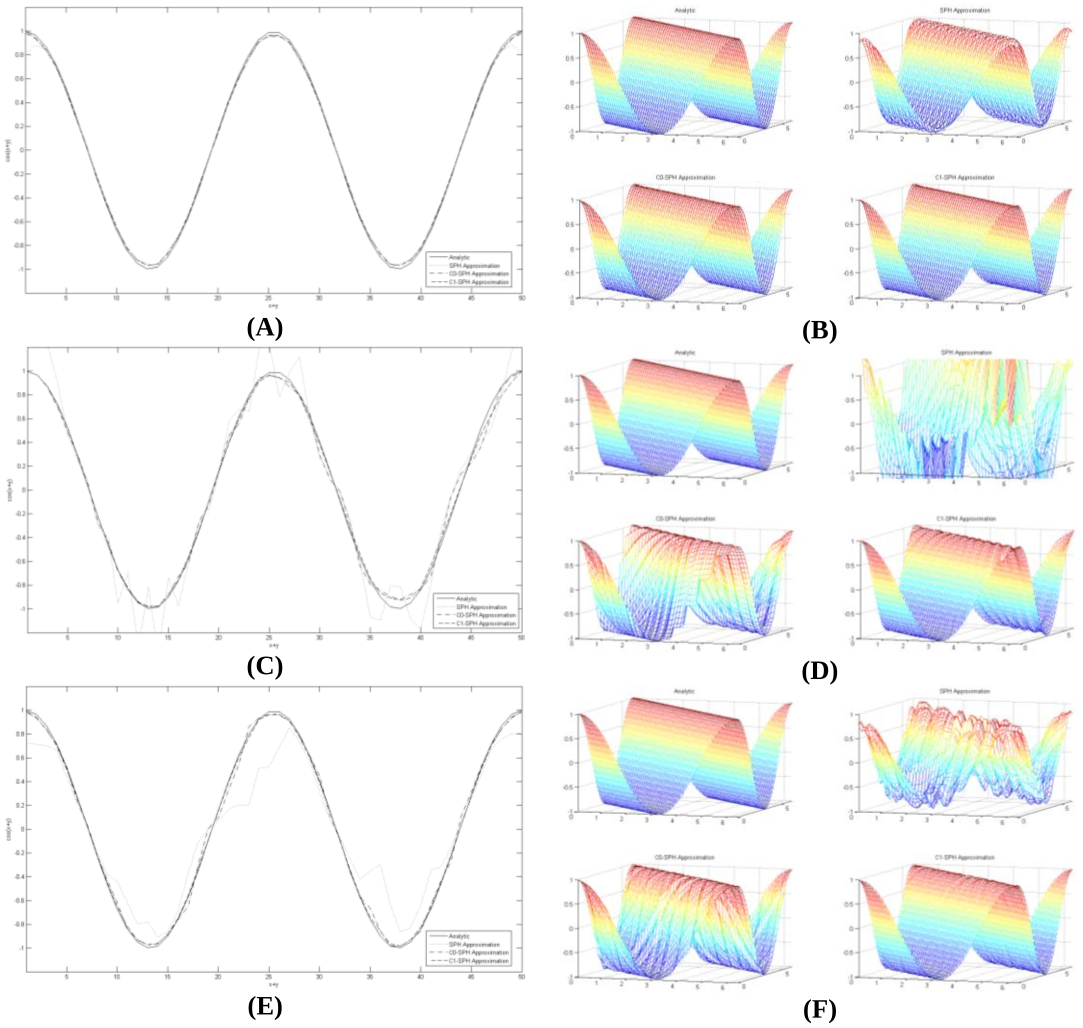



**Figure 2:** Comparison among the space profiles of the analytic 2D function $f(x, y) = \cos(x + y)$ and SPH simulations by using even (A, B), uneven (C, D) and random (E, F) distributions. Figures on the left side show the sections along the $x - y = 0$ plane.

| Algorithm | Even distribution | Uneven distribution | Random distribution |
|---|---|---|---|
| No Consistency | 0.0559 | 0.3961 | 0.3644 |
| 0-th order consistency | 0.0303 | 0.0737 | 0.0508 |
| 1-st order consistency | 0.0262 | 0.0409 | 0.0252 |

**Table 2:** root-mean-square errors obtained for the 2D function $f(x, y) = \cos(x + y)$.

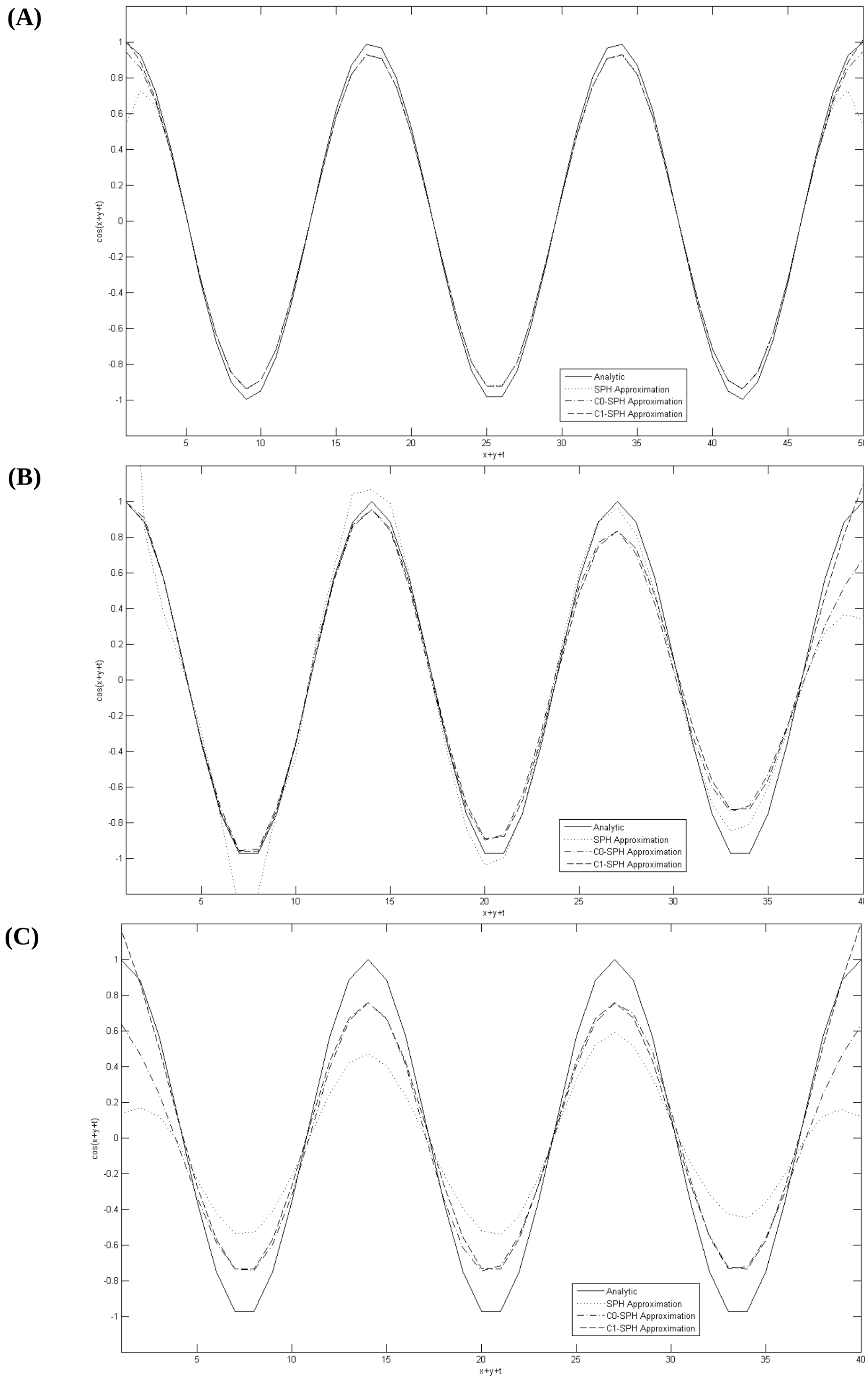



**Figure 3:** Comparison among the sections along the $x = y = t$ direction of the space profiles of the analytic 3D function $f(x, y, t) = \cos(x + y + t)$ and SPH simulations by using even (A), uneven (B) and random (C) distributions.

| Algorithm | Even distribution | Uneven distribution | Random distribution |
|---|---:|---:|---:|
| No Consistency | 0.1022 | 0.2995 | 0.5414 |
| 0-th order consistency | 0.0631 | 0.1291 | 0.2608 |
| 1-st order consistency | 0.0568 | 0.1167 | 0.2213 |

**Table 3:** root-mean-square errors obtained for the 3D function $f(x, y, t) = \cos(x + y + t)$.