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Variable annuities and embedded options valuation: a fuzzy logic framework

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Abstract

De Wit (1982) first applied fuzzy logic to insurance. That article sought to quantify the fuzziness in underwriting. Since then, the universe of discourse has expanded considerably and now also includes fuzzy logic applications involving classification, projected liabilities, future and present values, pricing, asset allocations and cash flows, and investments. The present contribution focuses on variable annuities valuation. As well known, with a VA contract owners are able to choose from a wide range of investment options called sub-accounts, enabling them to direct some assets into investment funds that can help keep pace with inflation and some into more conservative choices. Sub-accounts are similar to mutual funds that are sold directly to the public in that they invest in stocks, bonds and money market portfolios. Devine et al. (2004) coined the term "The New Variable Annuity" to highlight the introduction of guarantees, available as a rider feature to the overall product. Traditionally the guarantees were offered as a rider feature to the overall product package, but since 2000 insurance companies began offering more innovative guarantees, for an explicit price, as an optional choice to the customer. Aim of the present contribution is to price the options embedded in variable annuities contracts in a fuzzy logic framework.

1. Introduction

The goal of life insurance is to provide financial security to policyholders and their families. Traditionally, this security has been provided by means of a lump sum payable contingent on the death or survival of the insured life. The sum insured would be fixed and guaranteed. The policyholder would pay one or more premiums during the term of the contract for the right to the sum insured. But insurance markets around the world are changing. The public has become more aware of investment opportunities outside the insurance sector. Policyholder want to enjoy the benefits of equity investment in conjunction with mortality protection and insurers around the world have developed new insurance products to meet this challenge.

Among the proposed innovative product we can find the Variable Annuities (VA). According to the National Association of Variable Annuity Writers (NAVA) "with e VA contract owners are able to choose from a wide range of investment options called sub-accounts, enabling them to direct some assets into investment funds that can help keep pace with inflation and some into more conservative choices. Sub-accounts are similar to mutual funds that are sold directly to the public in that they invest in stocks, bonds and money market portfolios". Devine et al. (2004) coined the term "The New Variable Annuity" to highlight the introduction of guarantees, available as a rider feature to the overall product. Traditionally the guarantees were offered as a rider feature to the overall product package, but since 2000 insurance companies began offering more innovative guarantees, for an explicit price, as an optional choice to the customer. The guarantees offered generally fall into four classes: Guaranteed Minimum Death Benefits (GMDBs) that guarantee a return of the principal invested upon the death of the policyholder; Guaranteed Minimum Accumulation Benefits (GMABs) similar to GMDBs except that instead of the guarantees being contingent on the death of the insured, they typically bite on specified policy anniversaries or between specified dates if the policy is still in-force. If the guarantee is available at maturity they are called Guaranteed Minimum Maturity Benefits (GMMBs); Guaranteed Income Benefits (GMIBs) guarantee a minimum income stream (typically in the form of a life annuity) from a specified future point in time; Guaranteed Minimum Withdrawal Benefits (GMWBs) guarantee e minimum income stream trough regular withdrawals from the account balance.

VA have existed in USA since 1950s. NAVA report that the first variable annuity was issued in 1952 . VA are now also spreading across Europe. Some of the more significant and high profile launches have been AXA's in France, Germany, Spain, Italy and Belgium as well as ING's launches in Spain, Hungary and Poland. Generali's launch (December 2007) in Italy and Ergo's launch (February 2008) launch in Germany. This is in addition to the various launches by Aegon, Hartford, Metlife and Lincoln in the U.K.

Over the years, many practical and academic contributions have been offered for describing the VAs and the guarantees embedded. Recently, the academic literature has shown a fervent interest to the topic of VA (cfr. Bauer et al. (2006), Chen et al. (2008), Coleman et al. (2006), Dai (2008), Holz (2006), Milevsky and Panyagometh (2001), Milevsky and Posner (2001), Milevsky M.A and Promislow S.D (2001), Milevsky and Salisbury(2002)., Milevsky and Salisbury (2006), Nielsen and Sandmann (2003)). Aim of this paper is to focus on the pricing of options embedded in the VA contract resorting to fuzzy theory.

As well known the Black-Scholes model and the Cox- Ross- Rubinstein (CRR) model has been widely applied for computing the optimal warrant price. Referring to the results obtained by Li and Han (2009) we apply fuzzy set theory to the binomial tree option pricing model (CRR) to price the put option embedded in a GMMB guarantee of a variable annuity contract.

Taking the Knightian uncertainty of financial markets into consideration, the randomness and fuzziness of underlying should be evaluated by both probabilistic and fuzzy expectation.. Han and Li make use of parabolic fuzzy numbers to discuss the fuzzy binomial option pricing model with uncertainty of both randomness and fuzziness, and derive expression for the fuzzy risk neutral probabilities, along with fuzzy expression for option prices. As a consequence they obtain weighted intervals for the risk neutral probabilities and for the expected fuzzy option prices.

2. Variable annuity contract with a Guaranteed Minimum Maturity Benefit and Guaranteed Death Benefit

Let us consider a portfolio of VA contracts offering GMDB and GMMB guarantees and issued to C independent lives. Each insured pays a unique premium P and at time zero the Company receives the sum C*P.

Assuming that the insured pays and initial charge for general expenses computed as a percentage *c* of the premium, the Company invests the net premiums $C \cdot P' = C \cdot (P - P \cdot c)$ into a Fund and each insured can choose between different investment strategy. By virtue of the GMDB guarantee, if the insured does not survive at the end of the month *t*, the Company pays a sum equal to the maximum between the guaranteed and the fund value. On the other hand, by virtue of the GMMB, the guarantee is available at the maturity T if the insurer is still alive. A monthly management charge is paid by each insured.

The obligations the Company has to front for the GMDB at time $t \in 1,2,...,T$ are:

$$GMDB_t = N_D(t) \cdot Max[F_t, G_t] \tag{2.1}$$

At time T for the GMMB we have:

$$GMMB_T = N_S(T) \cdot Max[F_T, G_T] \tag{2.2}$$

being $N_D(t)$ the number of deaths in [*t-1, t*] and $N_S(T)$ the number of survivors at time *t*.

We assume that: $\{N_D(t)\}_{t=1}^{T} \cup \{N_S(T)\}$ is multinomial with parameters $\left(C; q_x, {}_{1|}q_x, \ldots, {}_{T\text{-}1|}q_x, {}_n p_x\right)$ being C the number of policies issued at time zero, ${}_{t|}q_x$ the probability that a life aged x dies in the (t+1)-th month after issue and ${}_n p_x$ is the probability that a life aged x at issue is alive at time n.

We assume that the guaranteed is computed according to a roll up guarantee and $G_t = P' \cdot e^{g \cdot t}$ with $t \in 1,2,...,T$ and *g* the monthly guaranteed rate.

As well known by means of the put decomposition principle, (2.1) and (2.2) can be rewritten as follows:

$$GMDB_t = N_D(t) \cdot (F_t + Max[0, G_t - F_t]) \tag{2.3}$$

with $t \in [1,2,...,T]$ and:

$$GMMB_t = N_S(t) \cdot (F_T + Max[0, G_T - F_T]) \tag{2.4}$$

Therefore it is possible to rewrite (2.1) and (2.2) as the sum of the fund value and the payoff of a put option with strike price equal to G_t with $t \in 1,2,...,T$ and G_T respectively.

The quantity which allows us to assess the liabilities connected to the offered guarantees is the put payoff because it is a measure of the Company’s obligations month by month.

Referring to the GMDB the cash flows are:

$$CF_t = N_D(t) \cdot Max[0, G_t - F_t] \qquad t \in [1,2,...,T] \tag{2.5}$$

On the other hand for the GMMB we have:

$$CF_T = N_S(T) \cdot Max[0, G_T - F_T] \tag{2.6}$$

Conditioning on being the expected number of deaths $E_D(t)$ equal to the actual number of deaths $N_D(t)$ we can write the expression for the liabilities function at time zero as the price of a portfolio of put options where the number of options, for each *t*, is determined by the expected number of deaths:

$$L_0 = \sum_{t=1}^{T} E_D(t) \cdot P_0(t) + E_S(t) \cdot P_0(T) \tag{2.7}$$

where $P_0(t)$ is the price, at time zero, of a put option with maturity *t*, underlying $S_0(t) = P' \cdot (1-m)^t$ and exercise price $G(t) = P' \cdot e^{g \cdot t}$.

Our interest now is in the evaluation of the loss function at time zero. On the basis of the preceding considerations we need to price the portfolio of embedded put options. To this aim we refer to a Binomial Tree option pricing model in a fuzzy logic framework.

2.1. Option Pricing and uncertainty

As stressed in the previous section, we are interest in assessing the liabilities connected to the guarantees that the Company offers in the contract. Of course we have to price the connected embedded options dealing with uncertainty that characterizes financial markets.

In asset pricing theory, uncertainty is modelled by means of state variables which play the role of sufficient statistics for the state of the world. The probability distributions, as well as the dynamic processes followed by the state variables, are assumed to be given and revealed to the agents in the economy.

Unfortunately in the real world distributions and stochastic dynamics are unknown or only partially known, and agent struggle to come by some hint about them. Usually this concept is referred to as information ambiguity, vagueness or uncertainty. Knight (1921) stressed that the distinction between risk (a situation in which the relative odds of the events are known) and uncertainty (a situation in which no such probability assignment can be done) was a key feature to explain investment decisions. We refer to such uncertainty as Knigthian uncertainty. Classical probability theory is incapable of accounting for this type of uncertainty.

Recently there has been a growing interest in using fuzzy numbers to deal with uncertainty. Many authors have tried to deal fuzziness along with randomness in option pricing models. For example Wu applied fuzzy approach to Black and Scholes formula. Zmeskal applied Black and Scholes methodology to appraise equity as a European call option. He used the input data in the form of fuzzy numbers to price the option. Carlson and Fuller use the possibility theory to fuzzy real option valuation.

Li and Han provide a fuzzy binomial model of option price determination in which the Knightan uncertainty plays a role. By modelling the underlying in each state of the world as a fuzzy number they obtain a possibility distribution on the risk neutral probability, i.e. a weighted interval of probability. By computing the option price under this measure they get a weighted expected value interval for the price and thus they are able to determine a 'most likely' option value within the interval. Moreover, by means of the so-called defuzzification procedure it is possible to associate to the option price a crisp number that summarizes all the information contained. They get an index of the fuzziness present in the option price, that tells us the degree of imprecision intrinsic in the model.

The information given by this kind of approach can be very useful to the Company's valuations, when pricing the options embedded into the contract to asses potential losses connected to the portfolio.

3. The traditional Binomial Tree Option Pricing Model

Let us start by describing the traditional Binomial option pricing model. Binomial option pricing model was proposed by Cox, Ross and Rubinstein (CRR) in 1979. Henceforth CRR model has a simple structure it is widely applied in the financial market and is one of the basic options pricing methods.

The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time. This is done by means of a binomial lattice (tree), for a number of time steps between the valuation and expiration date. Each node in the lattice represents a possible price of the underlying at a given point in time.

Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

Option valuation using this method is, as described, a three-step process: price tree generation, calculation of option value at each final node, sequential calculation of the option value at each preceding node.

The tree of prices is produced by working forward from valuation date to expiration. At each step, it is assumed that the underlying instrument will move up or down by a specific factor u or d respectively (where, by definition, $u \geq 1$ and $0 < d \leq 1$). So, if S is the current price, then in the next period the price will either be $S_{up} = S \cdot u$ or $S_{down} = S \cdot d$. The up and down factors are calculated using the underlying volatility, σ, and the time duration of a step, Δt, measured in years (using the day count convention of the underlying instrument). From the condition that the variance of the log of the price is $\sigma^2 \Delta t$, we have:

$$u = e^{\sigma \sqrt{\Delta t}} \tag{3.1}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \tag{3.2}$$

At each final node of the tree – i.e. at expiration of the option T – the option value is simply its intrinsic value that is Max [(S_T − K), 0] for a call option and Max [(K – S_T), 0], for a put option where K is the strike price and S_T is the spot price of the underlying asset at expiration time T. Once the above step is complete, the option value is then found for each node, starting at the penultimate time step, and working back to the first node of the tree (the valuation date) where the calculated result is the value of the option.

Let us consider a one-step example, that is the expiration date is T=2. We refer to e put option, being interested in this kind of derivative for our purposes. Suppose that the price of the underlying at period t=1 is S. The one step option pricing model can inference two possible stock prices (up and down movements) at some period t=2. Then we can calculate the put price at time 1.

Under the risk neutrality assumption, today's fair price of a derivative is equal to the expected value of its future payoff discounted by the risk free rate r and the discounting factor is $e^{-r \cdot \Delta t}$.

Therefore the expected value is calculated using the option values from the later two nodes (*Option up* and *Option down*) weighted by their respective probabilities; the probability p of an up move in the underlying and "probability" (1-p) of a down move. The two probabilities are respectively equal to:

$$p = \frac{a-d}{u-d} \qquad \text{and} \qquad 1-p = \frac{u-a}{u-d} \tag{3.3}$$

where $a = e^{r \cdot \Delta t}$

Let us define the following:

$$S_{up} = uS \tag{3.4}$$

$$S_{down} = uS \tag{3.5}$$

where S_{up} is the underlying price for the next period (at t=2) when it moves up; S_{down} is the underlying price for the next period (at t=2) when it goes down.

Moreover let

$$P_{up} = Max(K - S_{up})$$

$$P_{down} = Max(K - S_{down})$$

where P_{up} is the put value at t=2 in case of upward underlying and P_{down} is the put value at t=2 in case of downward underlying; K is the exercise price.

Finally we get the put price at t=1:

$$P = (p \cdot P_{up} + (1-p) \cdot P_{down}) \cdot e^{-r \cdot \Delta t} = \frac{(a-d) \cdot P_{up} + (u-a) \cdot P_{down}}{(u-d) \cdot a}$$

Similarly we can calculate the put value considering an higher time steps number.

4. The fuzzy Binomial Option Pricing model: the Fuzzy Binomial Tree

A fuzzy number A is known as a parabolic fuzzy number if there exist five parameters (*a*,*b*,*c*,*d*,*n*) and the membership function of A and the membership function of A is

$$\mu_A(x) = \begin{cases} 0, & x \le a \\ \left(\frac{x-a}{b-a}\right)^n, & a < x \le b \\ 1, & b < x \le c \\ \left(\frac{d-x}{d-c}\right)^n, & x > d \end{cases} \tag{4.1}$$

Parabolic type fuzzy number A will be denoted by $A = [a,b,c,d]_n$. If n=1, we simply write $A = [a,b,c,d]$, which is known as a trapezoidal fuzzy number. If $n \ne 1$, a fuzzy number $A^* = [a,b,c,d]_n$ is a modification of a trapezoidal fuzzy number $A = [a,b,c,d]$. If n>1, then A^* is a concentration of A. If 0<n<1, then A^* is a dilatation of A.

Alternatively the parabolic fuzzy number is defined in terms of its α-cuts by the following formula:

$$A(\alpha) = \left[a + \alpha^{1/n}(b-a), d\text{-}a^{1/n}(d-c)\right], \quad \alpha \in [0,1]$$

In order to introduce the fuzzy pricing methodology we refer to section 3 and we first consider a one period model, with $t \in [0,1]$. The assumption is that the price of underlying at t=1 takes only two possible values: given the current value S_0 it may either jump up or down with an exogenously given probability *p* and *(1-p)* with $p \in [0,1]$. Let *u* and *d* be the up and down crisp jump factors, respectively, the standard methodology leads to set $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u}$, where σ is the volatility of the underlying asset.

There are different methods for estimating the stock volatility either from historical data or from option prices. Since it is often hard to give an accurate estimate of the stock volatility, it may be convenient let it take interval values. As suggested by Avellaneda and Paras (1996), this is a way to incorporate heteroschedasticity (i.e. the volatility of volatility). Moreover it may be the case that not all the members of the interval have the same reliability, as central members are normally more likely than others.

Instead of modelling volatility as a fuzzy quantity, it is possible to model directly the up and down jump factors of the stock price.

5. The risk neutral probability intervals

Let us consider a one period model where the two basic securities are the money market account and the risky stock. The money market account is worth 1 at t=0 and its value at t=1 is 1+*r*, where *r* is the risk-free interest rate. The stock price at t=0, S_0 is observable while its price at t=1 is obtained by multiplying S_0 by the jump factors.

The standard methodology for deriving the risk neutral probabilities yields to the system

$$\begin{cases} p_d + p_u = 1 \\ \dfrac{d}{1+r} \cdot p_d + \dfrac{u}{1+r} \cdot p_u = 1 \end{cases} \tag{5.1}$$

where p_u is the risk neutral probability for the increase in the stock price and p_d is the risk-neutral probability for its decrease. Solving the system we have:

$$p_u = \frac{a-d}{u-d} \quad \text{and} \quad p_d = \frac{u-a}{u-d}$$

where $a = e^{r \cdot \Delta t}$.

In a fuzzy framework, the parameters *u* and *d* of equations 5.1 are parabolic fuzzy numbers denoted by $u = (u_1, u_2, u_3, u_4)_n$, $d = (d_1, d_2, d_3, d_4)_n$. The α-cut of *u* and *d* is

$$\begin{aligned} u(\alpha) &= \left[\underline{u}(\alpha), \overline{u}(\alpha)\right] \\ &= \left[u_1 + \alpha^{1/n}(u_2 - u_1), u_4 - \alpha^{1/n}(u_4 - u_3)\right], \quad \forall \alpha \in [0,1] \\ d(\alpha) &= \left[\underline{d}(\alpha), \overline{d}(\alpha)\right] \\ &= \left[d_1 + \alpha^{1/n}(d_2 - d_1), d_4 - \alpha^{1/n}(d_4 - d_3)\right], \quad \forall \alpha \in [0,1] \end{aligned} \tag{5.2}$$

By writing *u* and *d* in 5.1 in terms of α-cut we get

$$\begin{cases} p_d(\alpha) + p_u(\alpha) = 1 \\ \dfrac{[\underline{d}(\alpha), \overline{d}(\alpha)]}{1+r} p_d(\alpha) + \dfrac{[\underline{u}(\alpha), \overline{u}(\alpha)]}{1+r} p_u(\alpha) = [1,1] \end{cases} \tag{5.3}$$

It is easy to check that the following duality relations hold:

$$\underline{p}_d(\alpha)+\overline{p}_u(\alpha)=1, \qquad \underline{p}_u(\alpha)+\overline{p}_d(\alpha)=1$$

The system 5.1 can be split in the following two:

$$\begin{cases} \underline{p}_d(\alpha)+\overline{p}_u(\alpha)=1 \\ \dfrac{d_1+\alpha^{1\backslash n}(d_2-d_1)}{1+r}\underline{p}_d(\alpha)+\dfrac{u_1+\alpha^{1\backslash n}(u_2-u_1)}{1+r}\overline{p}_u(\alpha)=1 \end{cases} \tag{5.4}$$

$$\begin{cases} \overline{p}_d(\alpha)+\underline{p}_u(\alpha)=1 \\ \dfrac{d_4+\alpha^{1\backslash n}(d_4-d_3)}{1+r}\overline{p}_d(\alpha)+\dfrac{u_4+\alpha^{1\backslash n}(u_4-u_3)}{1+r}\underline{p}_u(\alpha)=1 \end{cases} \tag{5.5}$$

Solving system 5.4 we get:

$$\overline{p}_u(\alpha)=\frac{1+r-d_1-\alpha^{1\backslash n}(d_2-d_1)}{u_1-d_1+\alpha^{1\backslash n}(u_2+d_1-u_1-d_2)}$$

$$\underline{p}_d(\alpha)=\frac{-1-r+u_1+\alpha^{1\backslash n}(u_2-u_1)}{u_1-d_1+\alpha^{1\backslash n}(u_2+d_1-u_1-d_2)}$$

Solving system 5.5 we get:

$$\underline{p}_u(\alpha)=\frac{1+r-d_4-\alpha^{1\backslash n}(d_4-d_3)}{u_4-d_4+\alpha^{1\backslash n}(u_4-u_3-d_4+d_3)}$$

$$\overline{p}_d(\alpha)=\frac{-1-r+u_4-\alpha^{1\backslash n}(u_4-u_3)}{u_4-d_4-\alpha^{1\backslash n}(u_4-u_3-d_4+d_3)}$$

The two solutions represent the α-cut of the risk-neutral probability p_u and p_d :

$$p_u(\alpha)=[\underline{p}_u(\alpha),\overline{p_u}(\alpha)] \qquad p_d(\alpha)=[\underline{p}_d(\alpha),\overline{p_d}(\alpha)].$$

Differently from the standard binomial option pricing model, it is possible to obtain risk-neutral probability intervals instead of point values. This is clearly a consequence of the assumptions on the stock price.

The risk-neutral probability intervals arise from the ambiguity of the stock price at time t=1. Moreover the intervals of risk neutral probabilities are weighted, i.e. they are fuzzy numbers.

This is a very important feature of pricing options in a fuzzy framework, since it allows to find a weighted expected value interval for the option price.

6. The fuzzy option pricing

In this section we use the risk-neutral probabilities obtained in the previous section in order to price an option. Li and Han obtain some results on European call options.

Let us consider the date t=1. The stock price is given by either $S_0 d$ or $S_0 u$. Since *u* and *d* are parabolic fuzzy numbers, it follows that the stock price S_1 at t=1 in each state is represented by a parabolic fuzzy number. We denote the put payoff in state 'up' with *C(u)* and in state down with *C(d)*.

Applying the algebra of fuzzy numbers, we obtain the put payoff, which is still a parabolic fuzzy number equal to

$$\begin{aligned} C_u(\alpha) &= [\underline{C_u}(\alpha), \overline{C_u}(\alpha)] \\ &= [Max(S_0 \underline{u}(\alpha) - K, 0), Max(S_0 \overline{u}(\alpha) - K, 0)] \end{aligned}$$

We assume that the strike price is between the highest value of the stock in state down and the lowest value of the stock in state up: $S_0 d_4 \leq K \leq S_0 u_1$, then we get

$$\begin{aligned} C_u(\alpha) &= [\underline{C_u}(\alpha), \overline{C_u}(\alpha)] \\ &= [S_0(u_1 + \alpha^{1\backslash n}(u_2 - u_1)) - K, S_0(u_4 - \alpha^{1\backslash n}(u_4 - u_3)) - K] \end{aligned}$$

It is now possible to determine the call price C_0 by means of the risk-neutral valuation approach, as follows:

$$C_0 = \frac{1}{1+r}\hat{E}[C_1] = \frac{1}{1+r}[p_d C_d + p_u C_u]$$

where $\hat{E}$ stands for expectation under the risk-neutral probabilities and C_1 is the payoff of the put at t=1.

Since the call has zero payoff in the down state, the option pricing formula simplifies to

$$C_0 = \frac{1}{1+r}[p_u C_u]$$

The α-cut of C_0 is

$$C_0(\alpha)=[\underline{C_0}(\alpha),\overline{C_0}(\alpha)]$$
$$=\begin{bmatrix} \dfrac{S_0(u_1+\alpha^{1\backslash n}(u_2-u_1))-K}{1+r}\dfrac{1+r-d_4+\alpha^{1\backslash n}(d_4-d_3)}{u_4-d_4-\alpha^{1\backslash n}(u_4-u_3-d_4+d_3)}, \\ \dfrac{S_0(u_4+\alpha^{1\backslash n}(u_4-u_3))-K}{1+r}\dfrac{1+r-d_1+\alpha^{1\backslash n}(d_2-d_1)}{u_1-d_1-\alpha^{1\backslash n}(u_2+u_1-u_1-d_2)} \end{bmatrix}$$

It is easy to prove that α increases the call option interval of prices shrinks. If $u_2=u_3, d_2=d_3$ and α=1 the interval collapses into one single value.

6.1. A Multi Period Binomial Tree

Let us now extend the pricing methodology first to a two period and then to a multiple period binomial setting. We will restrict our attention to the case in which the up and down factors are the same at every stage. The stock price at t=1 in each state is represented by a parabolic fuzzy number, in particular

$$S_0u=(S_0u_1,S_0u_2,S_0u_3,S_0u_4)_n \qquad S_0d=(S_0d_1,S_0d_2,S_0d_3,S_0d_4)_n$$

At time t=2 each of S_0u and S_0d may further move either up or down as S_0uu , S_0ud, S_0dd, S_0du which are still fuzzy numbers and may be expressed as:

$$S_0dd=(S_0d_1^2,S_0d_2^2,S_0d_3^2,S_0d_4^2)_n$$
$$S_0ud=(S_0u_1d_1,S_0u_2d_2,S_0u_3d_3,S_0u_4d_4)_n$$

Generalizing this result, the stock price at each node of stage t may take t+1 possible values:

$$S_{i,.t}=S_0u^id^{t-i}, \qquad i=0,1,....,t$$

The possibility of taking each value $S_{t,i}$ is $\begin{pmatrix} t \\ i \end{pmatrix} p_u^i p_d^{t-i}$.

The stock price at each node of stage t will be equal to the following parabolic fuzzy number:

$$S_{t,i}=S_0u^id^{t-i}=(S_0{u_1}^i{d_1}^{t-i},S_0{u_2}^i{d_2}^{t-i},S_0{u_3}^i{d_3}^{t-i},S_0{u_4}^i{d_4}^{t-i})_n$$

The α-cut of $S_{t,i}$ is

$$S_{t,i}(\alpha)=S_0u^id^{t-i}(\alpha)=S_0[u_1^id_1^{t-i}+\alpha^{1\backslash n}(u_2^id_2^{t-i}-u_1^id_1^{t-i}),u_4^id_4^{t-i}-\alpha^{1\backslash n}(u_4^id_4^{t-i}-u_3^id_3^{t-i})]$$

As for the pricing of European call option in a t periods model, the call payoff at stage t has t+1 possible values (t=0,1,….,T):

$$C_{t,i} = Max(S_0 u^i d^{t-i} - K,0), \qquad i = 0,1,\ldots,t; \quad t = 0,1,\ldots,T.$$

The possibility of taking each value $C_{t,i}$ is $\binom{t}{i} p_u^i p_d^{t-i}$,

So we have

$$C_0 = \left(\frac{1}{1+r}\right)^T \hat{E}(C_T) = \left(\frac{1}{1+r}\right)^T \sum_{i=0}^{T} \binom{T}{i} p_u^i p_d^{T-i} C_{T,i}$$

Because $C_{t,i}$ is a fuzzy number, then

$$C_{t,i} = (S_{t,i} - 1_{\{K\}}) \vee 1_{\{0\}} = (1_{\{S_0\}} u^i d^{t-i} - 1_{\{K\}}) \vee 1_{\{0\}}, \qquad i = 0,1,\ldots.,t$$

Whose α-cut is:

$$\begin{aligned} C_{t,i}(\alpha) &= [\underline{C_{t,i}}(\alpha), \overline{C_{t,i}}(\alpha)] \\ &= [Max(S_0 u_1^i d_1^{t-i} + \alpha^{1\backslash n} S_0 (u_2^i d_2^{t-i} - u_1^i d_1^{t-i}) - K,0), \\ &\quad Max(S_0 u_4^i d_4^{t-i} + \alpha^{1\backslash n} S_0 (u_4^i d_4^{t-i} - u_3^i d_3^{t-i}) - K,0)] \end{aligned}$$

Through backward induction, we may get the payoff of European call option at stage t=0:

$$C_0 = \left(\frac{1}{1+r}\right)^T \left[\sum_{i=0}^{T} \binom{T}{i} \underline{p}_u^i(\alpha) \underline{p}_d^{T-i}(\alpha) \underline{C}_{T,i}(\alpha), \sum_{i=0}^{T} \binom{T}{i} \overline{p}_u^i(\alpha) \overline{p}_d^{T-i}(\alpha) \overline{C}_{T,i}(\alpha) \right]$$

6.2. The Defuzzification procedure and volatility estimation

For operative purposes, it may be convenient to find a crisp number that synthesizes the call option weighted interval. This type of problem is known in the literature as defuzzification procedure.

There are many methods (e.g. Cox 1994) that, depending on the kind of fuzzy number that we want to defuzzify, provide scalar that better represents the information contained.

Li and Han propose a method that is based on the intuitive idea that the best defuzzifier is the scalar that is closest to the fuzzy number in the following sense.

Define a metric D as the distance between a parabolic fuzzy number C and a crisp number *x*.

In order to find a scalar *x* that minimises the distance with the call price, we have to solve the following problem:

$$MinD(C,x)=\int_0^1(\underline{C}(\alpha)-x)^2 d\alpha+\int_0^1(\overline{C}(\alpha)-x)^2 d\alpha$$

From the first order condition we get

$$x^*=\frac{1}{2}\int_0^1(\underline{C}(\alpha)+\overline{C}(\alpha))d\alpha$$

x^* is the scalar that is closer to the left and right part of the call price. Once the value of the scalar *x* is determined, it is possible to compute the numerical value of the distance D

$$D(C,x)=\int_0^1(\underline{C}^2(\alpha)+\overline{C}^2(\alpha))d\alpha-2x^2$$

The defuzzification procedure leads

$$x^*=\frac{1}{2}(a+d)+\frac{1}{2}\frac{n}{n+1}(-a+b+c-d)$$

At this step, setting $\sigma=(\sigma_1,\sigma_2,\sigma_3,\sigma_4)_n$, $u_1=e^{\sigma_1\sqrt{\Delta t}}$, $u_2=e^{\sigma_2\sqrt{\Delta t}}$, $u_3=e^{\sigma_3\sqrt{\Delta t}}$, $u_4=e^{\sigma_4\sqrt{\Delta t}}$ $d_1=1/u_1, d_2=1/u_2, d_3=1/u_3, d_4=1/u_4$ they estimate the four volatility parameters by solving the following non linear optimization problem:

$$\min_{\sigma_1,\sigma_2,\sigma_3,\sigma_4,n}\sum_{t=1}^{T}(P_F(\sigma_1,\sigma_2,\sigma_3,\sigma_4,n)-P_M)^2$$
$$s.t.\begin{cases}\sigma_1<\sigma_2<\sigma_3<\sigma_4\\ -\sigma_1\sqrt{\Delta t}<r\Delta t<\sigma_1\sqrt{\Delta t}\end{cases}$$

where P_F is the defuzzified theoretical price (i.e. the crisp equivalent of the fuzzy price), P_M is the actual market price, n is the number of observations and r is the continuously compounded interest rate. The second constraint requires the no arbitrage condition to be fulfilled.

7. Conclusions and Future Directions

We are interested in pricing variable annuity guarantees of the typical VA contract described in section 2.

In the recent literature two stochastic approaches are implemented: the traditional actuarial approach which uses a 'real world' projection and the market consistent approach which typically uses a 'risk neutral' projection . In general pricing practice varies across different countries and companies.

We believe that the use of market consistent approach for pricing variable annuities guarantees is the most appropriate method for actuaries and companies today. This approach uses stochastic valuation techniques consistent with the pricing of options. This flexible methodology enables most product benefit and charging structures to be accommodated, and facilitates the calculation of risk exposures that can be used to construct and manage a dynamic hedge portfolio.

As well known the Black-Scholes model and the Cox- Ross- Rubinstein (CRR) model has been widely applied for computing the optimal warrant price and are typically used in insurance to price embedded options.

In asset pricing theory, uncertainty is modelled by means of state variables which play the role of sufficient statistics for the state of the world. The probability distributions, as well as the dynamic processes followed by the state variables, are assumed to be given and revealed to the agents in the economy.

Unfortunately in the real world distributions and stochastic dynamics are unknown or only partially known, and agent struggle to come by some hint about them. Usually this concept is referred to as information ambiguity, vagueness or uncertainty. Knight (1921) stressed that the distinction between risk (a situation in which the relative odds of the events are known) and uncertainty (a situation in which no such probability assignment can be done) was a key feature to explain investment decisions. We refer to such uncertainty as Knigthian uncertainty. Classical probability theory is incapable of accounting for this type of uncertainty.

Recently there has been a growing interest in using fuzzy numbers to deal with uncertainty. Many authors have tried to deal fuzziness along with randomness in option pricing models. For example Wu applied fuzzy approach to Black and Scholes formula. Zmeskal applied Black and Scholes methodology to appraise equity as a European call option. He used the input data in the form of fuzzy numbers to price the option. Carlson and Fuller use the possibility theory to fuzzy real option valuation.

Li and Han provide a fuzzy binomial model of option price determination in which the Knightan uncertainty plays a role. By modelling the underlying in each state of the world as a fuzzy number they obtain a possibility distribution on the risk neutral probability, i.e. a weighted interval of probability. By computing the option price under this measure they get a weighted expected value interval for the price and thus they are able to determine a 'most likely' option value within the interval. Moreover, by means of the so-called defuzzification procedure it is possible to associate to the option price a crisp number that summarizes all the information contained. They get an index of the fuzziness present in the option price, that tells us the degree of imprecision intrinsic in the model.

The information given by this kind of approach can be very useful to the Company's valuations, when pricing the options embedded into the contract.

Of course the topic of setting what price should the policyholder be charged for guarantee benefit is an important issue for actuaries and risk managers.

Moreover a suitable pricing technique is essential to asses potential losses connected to the portfolio.

On the other hand the complex hybrid equity and interest rate options embedded in variable annuity products present formidable hedging challenges for the insurers who write them.

Actuarial risks of policyholders behaviour complicate this problem further. Few insurers have developed complete liability valuation models integrating all these factors. Yet, growth in the VA markets requires not only comprehensive valuation models, but also a means to measure the prospective performance of different hedging programs around these risks, and a way to help insurers decide how they are going to hedge. Aim of this research is to deepen these issues in a fuzzy logic framework.

8. References

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